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Differential nonhomogeneous models for elastic randomly cracked solids

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Abstract

A differential scheme is developed to approximate the elastic behaviour of randomly cracked solids, accounting for possible locally nonhomogeneous distribution of the cracks. Certain inclusion or matrix spaces within the solids are modelled as forbidden regions for the cracks. At small to intermediate values of the crack density and proportion of forbidden regions, the effective elastic moduli of the models do not differ much from each other, but the differences become profound at higher values of those parameters: the effective moduli can be very small and large (toward those of the uncracked solids) depending upon the particular nonhomogeneous arrangements of the cracks. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Among the effective medium approximation schemes developed to estimate the elastic moduli of randomly inhomogeneous media, the differential scheme (Bruggeman, 1935; Brinkman, 1952; Roscoe, 1952; Salganik, 1973; Boucher, 1976; McLaughlin, 1977; Norris, 1985, 1989; Zimmerman, 1985; Hashin, 1988; Phan-Thien and Pham, 1997; Pham, 1998a,b) appears to be in certain advantage: It always corresponds to a certain exact geometry; therefore, it never violates mathematical requirements of the homogenization procedure, including the imposed bounds. The scheme makes use of the exact solution of the problem for dilute suspension of inclusions (Eshelby, 1957; Christensen, 1979) or cracks (Bristow, 1960; Budiansky and O'Connell, 1976; Kachanov, 1992) in an incremental procedure. One can imagine the possible formation picture of polydispersed cracks: because of high stresses, many small original cracks are formed independently and randomly at weak places within a loaded material that can relax the stress level around their immediate neighbourhoods. These cracks advance further until their stress-relaxed zones come into contact

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(so the cracks of the same size level can be considered as noninteracting at the first approximation). As the external loads continue to act, some larger and more local-favourably oriented cracks among the already existing ones may advance further and relax the stress level around them thus preventing the original neighbouring cracks from developing simultaneously. These larger cracks continue to develop on a larger scale until their influence zones come into contact with each other. This process can continue further leading to a polydispersed crack picture. Although such hierarchical geometries are highly idealistic, the scheme and its specific limit for multiphase composite called the self-consistent approximation (Hill, 1965; Norris, 1985; Pham, 1998a) has been used and attested also in practice to approximate the behaviour of many realistic nonhierarchical random mixtures. The scheme corresponds, at least, to some exact geometry, so it secures the approximation from possible contradictory results and a large disagreement with the observed behaviour of practical mixtures. There is some caution about the path dependence of the differential scheme. In our view, that nonuniqueness is an advantageous flexibility of the scheme. Practically random inhomogeneous media have very complicated structures with many possible arrangements of inhomogeneities, and therefore, one should not expect to have some effective medium scheme that can give unique and exact values of the effective moduli. Any practical approximation scheme can take into account, at most, only some of the main characteristics of the mixtures. The path dependence of the differential scheme should allow us to include some more specific features of a particular geometry to improve the approximation. In that case, the construction path should be based on physical sense, as we try to do in this work.

2. Differential models

We consider a combined random suspension of spherical inclusions having elastic moduli K_i, G_i of volume fraction v_s, and randomly oriented penny-shaped cracks of density $\epsilon = (1/V)\sum a^3$ (a is the crack radius V, volume of the representative element) in the matrix of an uncracked material having elastic moduli K, G. The elastic moduli of the random (cracked) mixture are denoted by K_c, G_c. For the differential scheme, we make use of dilute suspension results for spherical inclusions (Hill, 1965; Christensen, 1979; Phan-Thien and Pham, 1997) and penny-shaped cracks (Bristow, 1960; Gibiansky and Torquato, 1996). Presume that the spherical inclusions and the circular cracks are comparable on size scales (having the same size (radius) patterns).

The construction of the differential model starts with the basic uncracked matrix of moduli K, G. At each step of the procedure, we add proportionally infinitesimal amounts $v_s\,\Delta t$ ($\Delta t \ll 1$) of the inclusions (moduli K_i, G_i) and $\epsilon\,\Delta t$ of randomly oriented cracks of the same size scale into an already constructed mixture of the previous step, which contains volume fraction $v_s t$ of the inclusion phase and crack density ϵt (the parameter t increases from 0 to 1, as the differential scheme proceeds). The inclusions and cracks added at this step must be considerably greater in sizes to those that have been added previously, and they will see an effective medium, owing to their relative sizes. The new mixture can be considered as a dilute suspension of inclusions of volume fraction $v_s\,\Delta t/(1+v_s\,\Delta t)$ and cracks of density $\epsilon\,\Delta t/(1+v_s\,\Delta t)$ in a matrix of moduli $K_c(t), G_c(t)$. The effective moduli ($K_c + \mathrm{d}K_c, G_c + \mathrm{d}G_c$) of the new mixture are

$$\begin{aligned} K_c + \mathrm{d}K_c &= K_c + \frac{v_s\,\Delta t}{1+v_s\,\Delta t}\,\frac{(K_i - K_c)(3K_c + 4G_c)}{3K_i + 4G_c} - \frac{\epsilon\,\Delta t}{1+v_s\,\Delta t}\,\frac{4K_c^2(3K_c+4G_c)}{3G_c(3K_c+G_c)}, \\ G_c + \mathrm{d}G_c &= G_c + \frac{v_s\,\Delta t}{1+v_s\,\Delta t}\,\frac{(G_i - G_c)(G_c + G_{*c})}{G_i + G_{*c}} \\ &\quad - \frac{\epsilon\,\Delta t}{1+v_s\,\Delta t}\,\frac{16G_c(9K_c+4G_c)(3K_c+4G_c)}{45(3K_c+2G_c)(3K_c+G_c)}, \end{aligned} \tag{1}$$

where

$$G_{*c} = G_c \frac{9K_c + 8G_c}{6K_c + 12G_c}. \tag{2}$$

As the volume fraction of the inclusion phase, and crack density increase by

$$v_s \, \mathrm{d}t = \frac{v_s t + v_s \, \Delta t}{1 + v_s \, \Delta t} - v_s t = (1 - v_s t) \frac{v_s \, \Delta t}{1 + v_s \, \Delta t}, \tag{3}$$

$$\epsilon \, \mathrm{d}t = \frac{\epsilon t + \epsilon \, \Delta t}{1 + v_s \, \Delta t} - \epsilon t = (1 - v_s t) \frac{\epsilon \, \Delta t}{1 + v_s \, \Delta t}, \tag{4}$$

respectively, one deduces the following differential equations determining the effective elastic moduli of the mixture by the differential scheme

$$\begin{aligned}
&\frac{\mathrm{d}K_c}{\mathrm{d}t} = \frac{1}{1 - v_s t} \left[v_s \frac{(K_i - K_c)(3K_c + 4G_c)}{3K_i + 4G_c} - \epsilon \frac{4K_c^2(3K_c + 4G_c)}{3G_c(3K_c + G_c)} \right], \\
&\frac{\mathrm{d}G_c}{\mathrm{d}t} = \frac{1}{1 - v_s t} \left[v_s \frac{(G_i - G_c)(G_c + G_{*c})}{G_i + G_{*c}} - \epsilon \frac{16G_c(9K_c + 4G_c)(3K_c + 4G_c)}{45(3K_c + 2G_c)(3K_c + G_c)} \right], \\
&0 \leqslant t \leqslant 1, \quad K_c(0) = K, \quad G_c(0) = G.
\end{aligned} \tag{5}$$

In our composite model, the inclusion phase is tough and does not permit the cracks of the same size scale to enter it. Here, we are interested in a particular case when the uncracked material is elastically homogeneous, that means $K_i = K$, $G_i = G$. For some particulate aggregates (e.g. some rocks), the cracks are more favourable to originate at the weak boundary zones leaving the internal parts of the grains intact. These forbidden zones for the cracks, here, are modelled as spherical inclusions. Another possible cause for the local inhomogeneous distribution of the cracks is for a body with random cracks of the same size, the regions immediately near the middle of the two faces of a crack are stress-relaxed, so the neighbouring cracks might be oriented more favourably toward its perimeter region of high stress than to those relaxed ones. The inhomogeneity is related to the high degree of anisotropy of a crack with one from its three dimensions approaches zero. Such a material should behave like that with some fictitious forbidden zones for the cracks. Hence, presuming the cracks and the forbidden zones are on the same size scale, from Eq. (5), we obtain the equations (here $K_i = K$, $G_i = G$)

$$\begin{aligned}
&\frac{\mathrm{d}K_c}{\mathrm{d}t} = \frac{1}{1 - v_s t} \left[v_s \frac{(K - K_c)(3K_c + 4G_c)}{3K + 4G_c} - \epsilon \frac{4K_c^2(3K_c + 4G_c)}{3G_c(3K_c + G_c)} \right], \\
&\frac{\mathrm{d}G_c}{\mathrm{d}t} = \frac{1}{1 - v_s t} \left[v_s \frac{(G - G_c)(G_c + G_{*c})}{G + G_{*c}} - \epsilon \frac{16G_c(9K_c + 4G_c)(3K_c + 4G_c)}{45(3K_c + 2G_c)(3K_c + G_c)} \right], \\
&0 \leqslant t \leqslant 1, \quad K_c(0) = K, \quad G_c(0) = G.
\end{aligned} \tag{6}$$

In the case of absence of forbidden zones, Eq. (6) reduce to the usual ones of the differential scheme for cracked solids (Salganik, 1973; Zimmerman, 1985; Hashin, 1988). For numerical illustrations, we take $\epsilon = 0 \to 0.8$, v_s between 0.2 and 0.8, $v = 0.3$ $[K = 2G(1 + v)/3(1 - 2v)]$. The respective solutions of the Eq. (6) are represented in Tables 1 and 2. We see that as the crack density ϵ increases, the elastic moduli decreases considerably. With increasing proportion of forbidden zones v_s, the shear modulus G_c also decreases but slightly, while the bulk modulus K_c is almost unaffected by v_s in those ranges.

More intricately the forbidden zones can be of a shape other than spherical. We take the extreme case of (randomly oriented) platelet inclusions (volume proportion v_p, (Pham, 1998a). Then, instead of Eq. (6), the respective differential equations would take the form:

Table 1
Differential approximation for the bulk modulus K_c/K of cracked material with forbidden spherical zones

ϵ	$v_s = 0$	$v_s = 0.2$	$v_s = 0.4$	$v_s = 0.6$	$v_s = 0.8$
0.08	0.740	0.741	0.742	0.743	0.744
0.16	0.568	0.569	0.571	0.573	0.576
0.24	0.446	0.448	0.451	0.454	0.458
0.32	0.357	0.359	0.362	0.366	0.370
0.40	0.289	0.292	0.295	0.299	0.303
0.48	0.237	0.240	0.243	0.246	0.249
0.56	0.196	0.199	0.201	0.204	0.206
0.64	0.164	0.166	0.168	0.170	0.171
0.72	0.137	0.139	0.141	0.142	0.142
0.80	0.116	0.117	0.118	0.119	0.117

Table 2
Differential approximation for the shear modulus G_c/G of cracked material with forbidden spherical zones

ϵ	$v_s = 0$	$v_s = 0.2$	$v_s = 0.4$	$v_s = 0.6$	$v_s = 0.8$
0.08	0.894	0.894	0.894	0.894	0.893
0.16	0.796	0.796	0.795	0.795	0.794
0.24	0.707	0.706	0.705	0.704	0.702
0.32	0.626	0.624	0.623	0.620	0.617
0.40	0.552	0.551	0.548	0.545	0.541
0.48	0.487	0.484	0.481	0.477	0.471
0.56	0.428	0.425	0.421	0.417	0.409
0.64	0.376	0.372	0.368	0.362	0.353
0.72	0.329	0.325	0.321	0.314	0.303
0.80	0.288	0.284	0.279	0.271	0.259

$$
\begin{aligned}
&\frac{\mathrm{d}K_c}{\mathrm{d}t} = \frac{1}{1 - v_p t}\left[v_p \frac{(K - K_c)(3K_c + 4G)}{3K + 4G} - \epsilon \frac{4K_c^2(3K_c + 4G_c)}{3G_c(3K_c + G_c)}\right],\\
&\frac{\mathrm{d}G_c}{\mathrm{d}t} = \frac{1}{1 - v_p t}\left[v_p \frac{(G - G_c)(G_c + G_*)}{G + G_*} - \epsilon \frac{16G_c(9K_c + 4G_c)(3K_c + 4G_c)}{45(3K_c + 2G_c)(3K_c + G_c)}\right],\\
&0 \leqslant t \leqslant 1, \quad K_c(0) = K, \quad G_c(0) = G, \quad G_* = G\frac{9K + 8G}{6K + 12G}.
\end{aligned} \tag{7}
$$

For numerical illustrations, we take the same ranges for ϵ and v_p as was taken for previous spherical inclusion case. The results are reported in Tables 3 and 4. As in the previous example, the elastic moduli decrease considerably as the crack density ϵ increases. However, the effective moduli increase at increasing values of the proportion of forbidden platelet inclusions v_p.

3. Two-level models

The crack + spherical-forbidden zone model of the previous section was constructed with the presumption that the cracks and the inclusions are comparable on sizes. In the other case, when the cracks situated on the boundary of the grains within an aggregate are much smaller in sizes compared with those of the forbidden inclusions, it should be more appropriate to substitute the model by a two-level one. At first, we apply the differential scheme to estimate the moduli $\bar{K}, \bar{G}$ of the material having moduli K, G with crack density $\epsilon/(1 - v_s)$. Then, we again use the scheme to determine the moduli K_c, G_c of the composite

Table 3
Differential approximation for the bulk modulus K_c/K of cracked material with forbidden platelet zones

ϵ	$v_p = 0$	$v_p = 0.2$	$v_p = 0.4$	$v_p = 0.6$	$v_p = 0.8$
0.08	0.740	0.741	0.742	0.743	0.745
0.16	0.568	0.569	0.572	0.575	0.580
0.24	0.446	0.449	0.453	0.459	0.467
0.32	0.357	0.361	0.367	0.375	0.385
0.40	0.289	0.295	0.303	0.312	0.324
0.48	0.237	0.244	0.253	0.264	0.278
0.56	0.196	0.204	0.214	0.226	0.241
0.64	0.164	0.172	0.183	0.196	0.212
0.72	0.137	0.147	0.158	0.172	0.189
0.80	0.116	0.126	0.138	0.152	0.169

Table 4
Differential approximation for the shear modulus G_c/G of cracked material with forbidden platelet zones

ϵ	$v_p = 0$	$v_p = 0.2$	$v_p = 0.4$	$v_p = 0.6$	$v_p = 0.8$
0.08	0.894	0.894	0.894	0.894	0.893
0.16	0.796	0.796	0.796	0.795	0.795
0.24	0.707	0.706	0.706	0.706	0.706
0.32	0.626	0.625	0.625	0.625	0.626
0.40	0.552	0.552	0.553	0.554	0.557
0.48	0.487	0.487	0.489	0.491	0.496
0.56	0.428	0.429	0.432	0.436	0.444
0.64	0.376	0.378	0.382	0.388	0.398
0.72	0.329	0.333	0.339	0.347	0.360
0.80	0.288	0.293	0.300	0.311	0.326

with proportion v_s of forbidden spherical inclusions having moduli K, G suspended in the matrix of moduli $\bar{K}, \bar{G}$. The general solution of Eq. (5) is denoted as

$$K_c = S_K\{\epsilon, v_s, K_i, G_i, K, G\}, \qquad G_c = S_G\{\epsilon, v_s, K_i, G_i, K, G\}. \tag{8}$$

The effective moduli K_c, G_c of our two-level model are then determined by

$$\begin{aligned} \bar{K} &= S_K\{\epsilon/(1-v_s), 0, 0, 0, K, G\}, \qquad \bar{G} = S_G\{\epsilon/(1-v_s), 0, 0, 0, K, G\}, \\ K_c &= S_K\{0, v_s, K, G, \bar{K}, \bar{G}\}, \qquad G_c = S_G\{0, v_s, K, G, \bar{K}, \bar{G}\}. \end{aligned} \tag{9}$$

Consider also the opposite case: because of some strengthening process, the boundary regions of the grains become forbidden regions for the cracks, so the cracks should be concentrated in the internal central part of the grains. Presume also that the cracks are also of a much smaller size than the grains, so we should construct a two-level model for the case: At first, we apply the differential scheme to estimate the moduli $\bar{K}, \bar{G}$ of the material having virgin moduli K, G with crack density ϵ/v_s. Then again use the scheme to determine the moduli K_c, G_c of the composite with proportion v_s of spherical inclusions having moduli $\bar{K}, \bar{G}$ suspended in the matrix of moduli K, G (Note that in the considered model, the forbidden matrix region has the volume proportion $1 - v_s$). In particular, according to Eq. (8), we have the effective moduli K_c, G_c of our two-level model determined by

$$\begin{aligned} \bar{K} &= S_K\{\epsilon/v_s, 0, 0, 0, K, G\}, \qquad \bar{G} = S_G\{\epsilon/v_s, 0, 0, 0, K, G\}, \\ K_c &= S_K\{0, v_s, \bar{K}, \bar{G}, K, G\}, \qquad G_c = S_G\{0, v_s, \bar{K}, \bar{G}, K, G\}. \end{aligned} \tag{10}$$

The results of the calculations for two-level inclusion-forbidden model of Eq. (9) and matrix forbidden model (10) at some values of v_s and at crack density $\epsilon = 0 \rightarrow 0.8$ ($v = 0.3$) compared with the usual differential crack model without forbidden zones are plotted in Figs. 1 and 2. One can see that the moduli of the model (9) are smaller than those of the usual differential model. In particular, the model can have very small elastic moduli toward zero values at given positive crack density ϵ, when $v_s \to 1$. On the contrary, the moduli of the model (10) are larger and can approach those of the uncracked material at the given positive crack density ϵ, when $v_s \to 0$. Thus, the cracks distributed at the grain boundary are much more deteri-

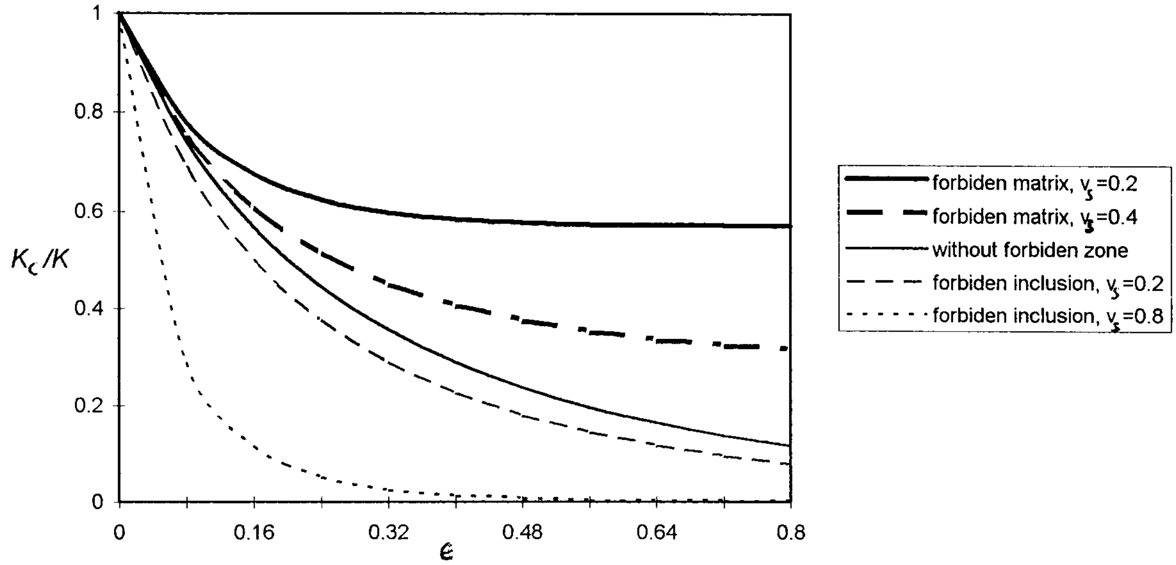

Fig. 1. Bulk modulus of two-level models compared with that of the usual differential model ($\nu = 0.3$).

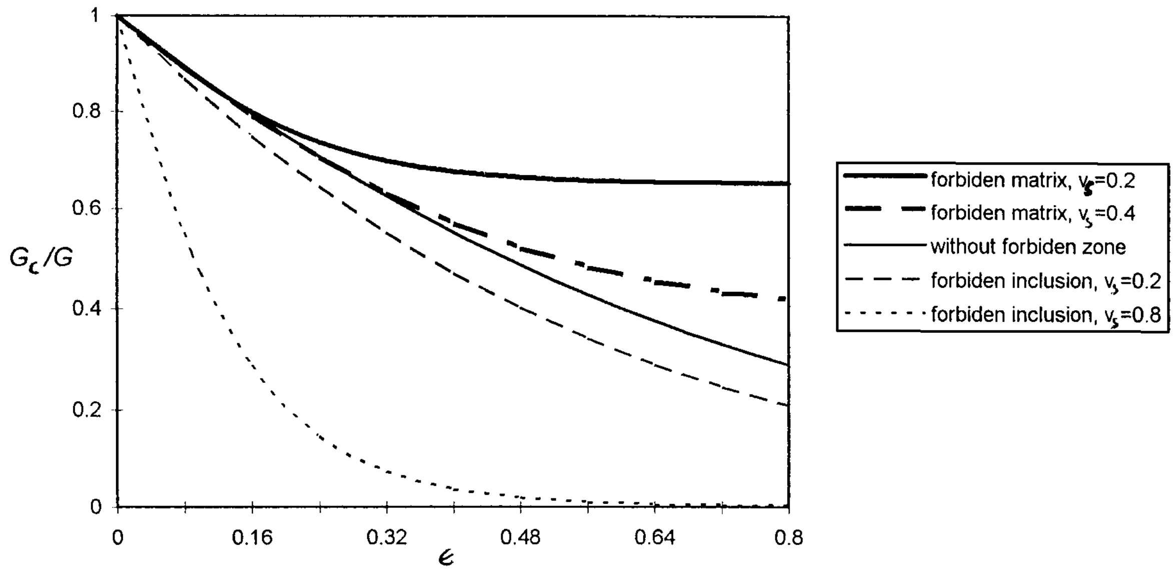

Fig. 2. Shear modulus of two-level models compared with that of the usual differential model ($\nu = 0.3$).

oratively-effective to the aggregate elastic moduli than those concentrated in the isolated centres of the grains.

4. Cracked porous material

Consider a cracked porous material with density ϵ of penny-shaped cracks and proportion v_s (porosity) of pores of spherical shape. Presume, that at first the pores and the cracks are on the same size scales. Substitute $K_i = 0, G_i = 0$ into the Eq. (5), we get the equations determining the elastic moduli K_c, G_c of our cracked porous model:

$$\frac{dK_c}{dt} = -\frac{1}{1 - v_s t}\left[v_s \frac{K_c(3K_c + 4G_c)}{4G_c} + \epsilon \frac{4K_c^2(3K_c + 4G_c)}{3G_c(3K_c + G_c)}\right],$$
$$\frac{dG_c}{dt} = -\frac{1}{1 - v_s t}\left[v_s \frac{G_c(G_c + G_{*c})}{G_{*c}} + \epsilon \frac{16G_c(9K_c + 4G_c)(3K_c + 4G_c)}{45(3K_c + 2G_c)(3K_c + G_c)}\right],$$
$$0 \leqslant t \leqslant 1, \quad K_c(0) = K, \quad G_c(0) = G; \tag{11}$$

or according to Eq. (8)

$$K_c = S_K\{\epsilon, v_s, 0, 0, K, G\}, \qquad G_c = S_G\{\epsilon, v_s, 0, 0, K, G\}. \tag{12}$$

The solutions of Eq. (11) at $v_s = 0, 0.2, 0.4, 0.6$ and $\epsilon = 0.08 \to 0.8$ ($v = 0.3$) are plotted in Figs. 3 and 4.

Now, consider the case when the cracks are on the much smaller size scales than those of the pores. For which, we use a two-level model: At first, we apply the differential scheme to estimate the moduli $\bar{K}, \bar{G}$ of the material having virgin moduli K, G with crack density $\epsilon/(1 - v_s)$. We then again use the scheme to determine the moduli K_c, G_c of the composite with pores of porosity v_s suspended in the matrix of moduli $\bar{K}, \bar{G}$. In particular, according to Eq. (8), we have

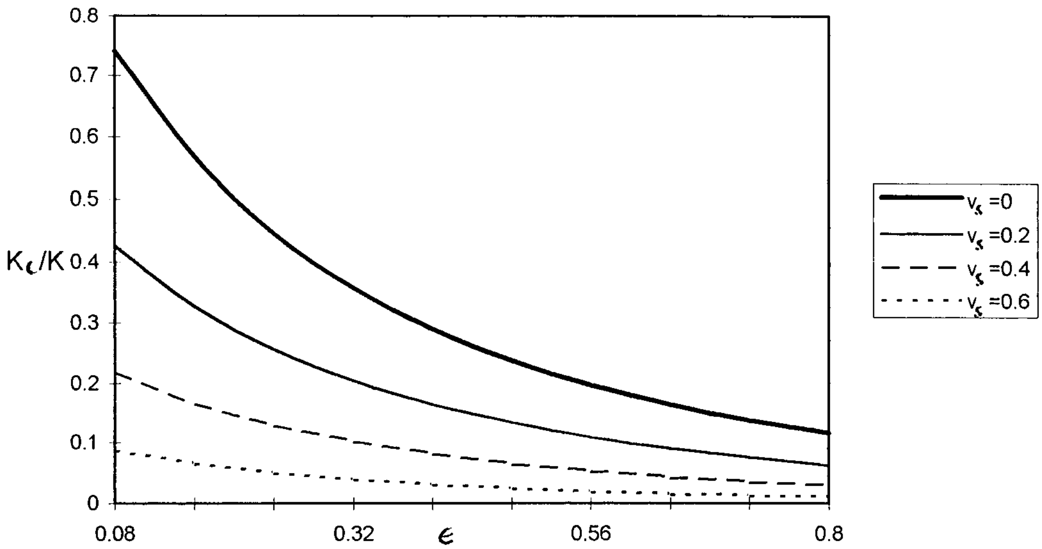

Fig. 3. Bulk modulus of the cracked porous material ($v = 0.3$).

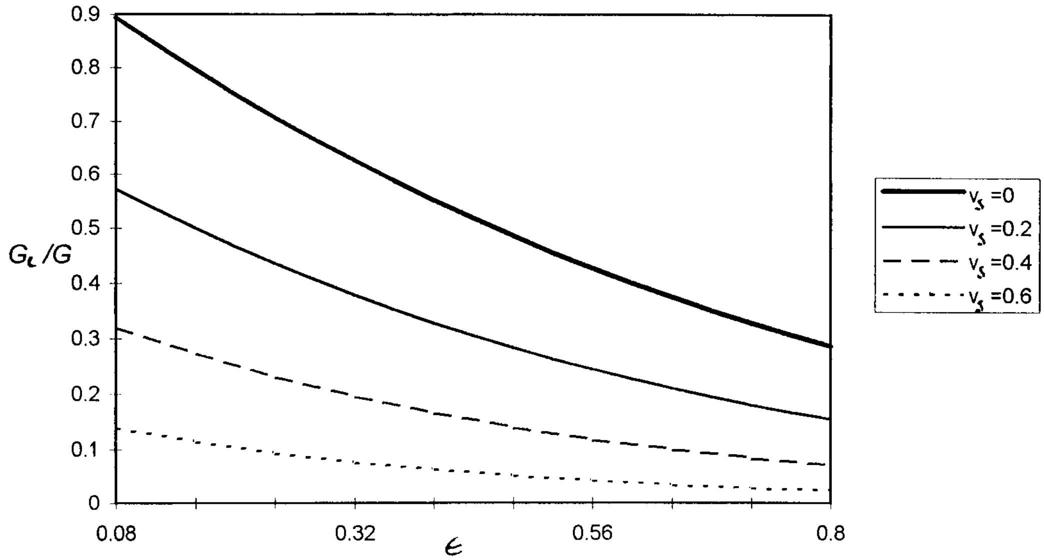

Fig. 4. Shear modulus of the cracked porous material ($v = 0.3$).

$$\bar{K} = S_K\{\epsilon/(1 - v_s), 0, 0, 0, K, G\}, \qquad \bar{G} = S_G\{\epsilon/(1 - v_s), 0, 0, 0, K, G\},$$
$$K_c = S_K\{0, v_s, 0, 0, \bar{K}, \bar{G}\}, \qquad G_c = S_G\{0, v_s, 0, 0, \bar{K}, \bar{G}\}. \tag{13}$$

Next, consider the case when the pores have much smaller sizes than those of the cracks. The two-level model for it is as follows: At first, we apply the differential scheme to estimate the moduli $\bar{K}, \bar{G}$ of the suspension of pores of porosity v_s in matrix having moduli K, G; Then, again use the scheme to determine the moduli K_c, G_c of the cracked material with virgin moduli $\bar{K}, \bar{G}$ and crack density ϵ. In particular, according to Eq. (8), we have

$$\bar{K} = S_K\{0, v_s, 0, 0, K, G\}, \quad \bar{G} = S_G\{0, v_s, 0, 0, K, G\},$$
$$K_c = S_K\{\epsilon, 0, 0, 0, \bar{K}, \bar{G}\}, \quad G_c = S_G\{\epsilon, 0, 0, 0, \bar{K}, \bar{G}\}. \tag{14}$$

The elastic moduli curves of the two-level models (13) and (14) compared with that of the one-level model (12) at $v_s = 0.2$ and $\epsilon = 0.08 \to 0.8$ ($v = 0.3$) are given in Figs. 5 and 6. We see that at the same crack density ϵ and porosity v_s the two-level small-pore large-crack model has the largest moduli, while the two-level small-crack large-pore model possesses the smallest ones. The one-level pore-crack model gives the intermediate moduli. However, the differences are not large.

5. Conclusions

The differential scheme has been developed to model the elastic behaviour of randomly cracked materials. An advantage of the scheme is that it corresponds, at least, to an exact geometry, while being used to estimate the elastic moduli of usual random mixtures as other effective medium approximate schemes do. The flexibility of the scheme allows us to take into account, beside the crack density, the shapes of the cracks as well as the possible nonhomogeneous crack arrangements or differences in the relative sizes of the inhomogeneities; the latter is the focus of this study. The nonhomogeneous distributions of the cracks have been included by introducing forbidden regions for the cracks within the base material. The differences in

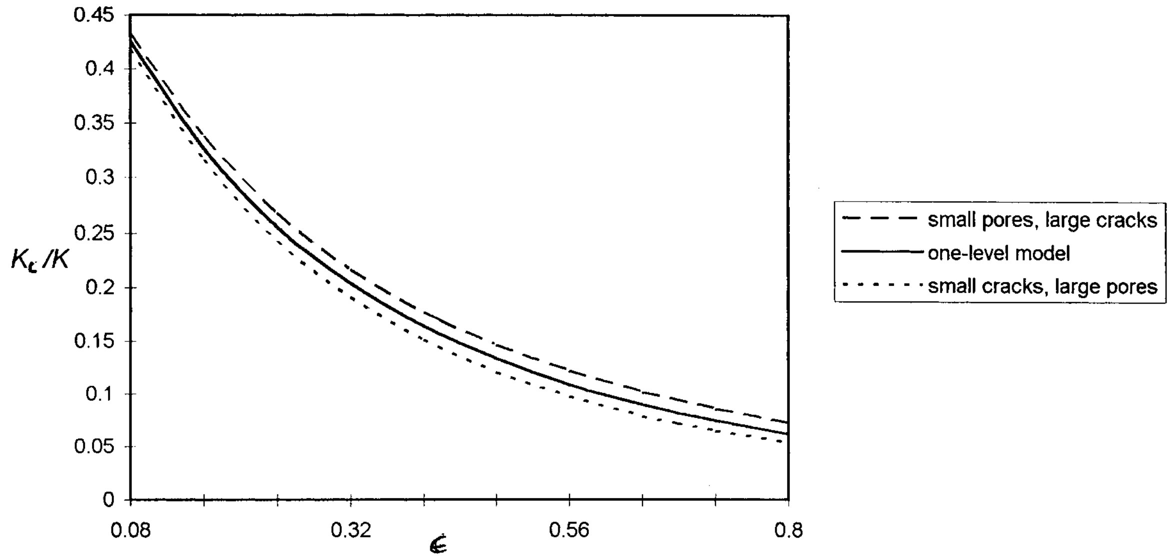

Fig. 5. Bulk modulus of the two-level cracked porous models ($\nu = 0.3$).

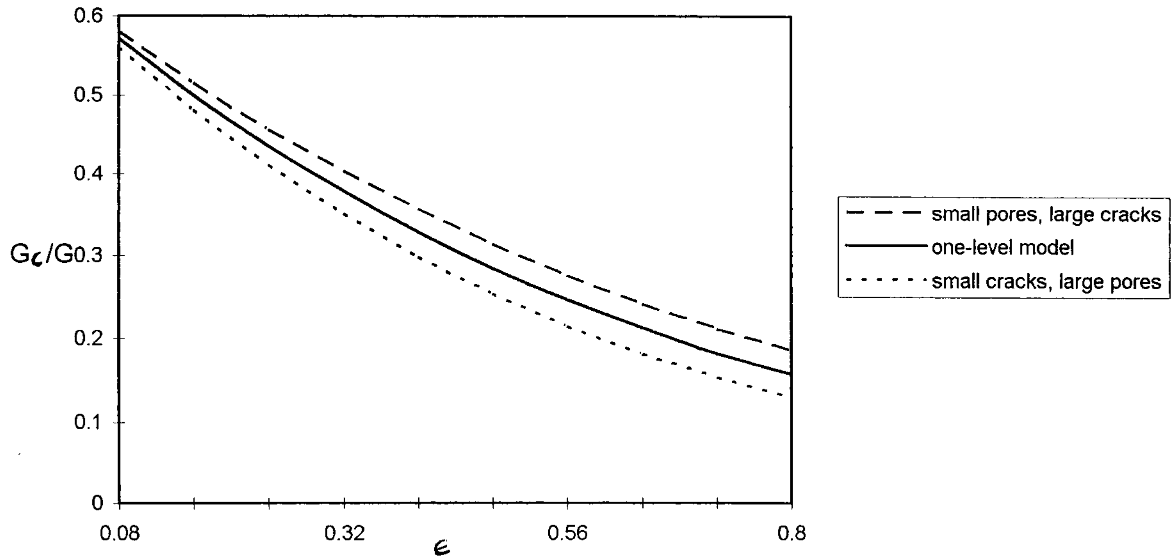

Fig. 6. Shear modulus of the two-level cracked porous models ($\nu = 0.3$).

the size scales of the inhomogeneities have been accounted for by using multi-scale differential scheme. The modified models may not differ much from each other at small to intermediate values of crack density and proportion of forbidden regions, but the differences may become large at higher values of those parameters. Whenever the differences appear large, the refined geometric features of a particular mixture cannot be ignored but to be accounted for a meaningful evaluation of effective moduli of the mixture. For random particulate aggregates, the small cracks (compared with the grain sizes) distributed at the grain boundary appear much more deteriorable to the aggregate effective moduli than those concentrated in the isolated centers of the grains. We expect that there would be no universal model and approximation for cracked bodies, but the specific ones. Within our approach, a construction path of the differential scheme should be based on physical sense from particular features of a crack configuration, to which an approximation should be restricted to.

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